Comparison of phase-aberrated laser beam quality criteria

Eugeny Perevezentsev, Anatoly Poteomkin, and Efim Khazanov

With any form of phase distortions there is the need to qualitatively characterize beam quality. Three different qualitative criteria are most commonly used for this purpose, each of them describing the beam with one ratio: the overlapping integral, the Strehl ratio, and the M^2 parameter. We have analyzed the interrelation of the above-mentioned criteria in the three most common types of beam quality degradation: thermal lens, electronic self-focusing, and spherical aberration. Approximate analytical expressions for all three criteria and three types of beam distortion are derived for Gaussian and super-Gaussian intensity profiles. The efficiency of characterizing those beams by various criteria is discussed.

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1. Introduction

Today the power of lasers with diffraction beam quality is on a significant rise. However, it is not easy to achieve ideal quality because of some parasitic effects and deteriorating laser beam quality. A beam changes differently under different mechanisms of distortions. In any case, it is necessary to characterize the beam quality quantitatively, with only one number. Three quantitative criteria are most frequently used: the overlapping integral χ, the Strehl ratio S, and the M^2 parameter.

By definition the overlapping integral χ is

$$\chi = \frac{\left| \iint\limits_{\Omega} E_{in} E_{out}{}^{*} \mathrm{d}S \right|^2}{\iint\limits_{\Omega} |E_{out}|^2 \mathrm{d}S \iint\limits_{\Omega} |E_{in}|^2 \mathrm{d}S}, \quad (1)$$

where Ω is the area of cross section of a beam, and E_{in} and E_{out} are the field distributions at the input and output of an aberrating element. The physical meaning of χ can be explained as follows. Let us represent the field at the output as a sum,

$$E_{out}(r) = NE_{in}(r) + E_0(r), \quad (2)$$

with fields E_{in}, (r) and $E_0(r)$ being orthogonal, i.e.,

$$\iint\limits_{\Omega} E_{in} E_0{}^{*} \mathrm{d}S = 0. \quad (3)$$

Then

$$\iint\limits_{\Omega} |E_{out}|^2 \mathrm{d}S = |N|^2 \iint\limits_{\Omega} |E_{in}|^2 \mathrm{d}S + \iint\limits_{\Omega} |E_0|^2 \mathrm{d}S, \quad (4)$$

with

$$\chi = |N|^2. \quad (5)$$

Thus as expressions (4) and (5) show, χ characterizes a portion of power of the undistorted (input) field in the distorted (output) field. The expressions for χ were derived in the case of self-induced thermal lensing of Gaussian[1–3] and super-Gaussian beams.[1]

By definition[4,5] the Strehl ratio S is equal to the ratio of intensities of the distorted beam $I_{F\,out}(0)$ and the undistorted beam $I_{F\,in}(0)$ in the focal plane of a

E. Perevezentsev (jeka@mail.nnov.ru), A. Poteomkin (ptmk@appl.sci-nnov.ru), and E. Khazanov are with the Institute of Applied Physics, Russia.

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lens on the optical axis:

$$S = \frac{I_{F\,out}(0)}{I_{F\,in}(0)}. \tag{6}$$

The physical meaning of the Strehl ratio comes from its definition: S indicates a decrease in intensity on the beam axis in the focal plane.

The influence of amplitude and phase distortions on the Strehl ratio has been studied in a number of works. For primary aberration, approximated equations for S were found for flattop beams[6] amplitude-weighted pupils,[7–10] and Gaussian pupils.[11,12] Beams with random aberration were studied in Refs. 13 and 14. In Ref. 15 the authors investigated the Strehl ratio decrease attributable to heat deposition in the laser active element. The Strehl ratio was also used to characterize the distortions of the beam passed through a channel flow of a He–N_2 turbulent mixing layer[16] and a beam focused into biaxially birefringent media.[17]

The M^2 parameter was suggested by Siegman[18] in 1990 and is now widely used to characterize laser beam quality. We will consider axially symmetric beams. In this case

$$M^2 = 2\pi\sigma_0\sigma_f, \tag{7}$$

where σ_0 and σ_f are the second central moments of the field intensity and the spatial spectrum of the field, respectively:

$$\sigma_0^{\,2} = \frac{\int_0^\infty |E(r)|^2 r^3 \mathrm{d}r}{\int_0^\infty |E(r)|^2 r \mathrm{d}r}, \qquad \sigma_f^{\,2} = \frac{\int_0^\infty \left|\widehat{u}(k)\right|^2 k^3 \mathrm{d}k}{\int_0^\infty \left|\widehat{u}(k)\right|^2 k \mathrm{d}k}, \tag{8}$$

where σ_0 is calculated in the beam waist, $\widehat{u}(k)$ is the distribution of the spatial spectrum of the field, and $k = 2\pi/\lambda$ is a wavenumber. In Refs. 19 and 20 the M^2 parameter is defined by an 86.5% power-content radius, but we will follow the most commonly used definition in expression (7).

By calculating the M^2 parameter from expressions (7) and (8), it is necessary to find the second moments of the far-field intensity. This can be done easily for the beams with a plane phase.[21–23] In a more general and complicated case of phase aberration, some examples were studied in Refs. 24 and 25.

A remarkable feature of the beam size determined as a second moment is its hyperbolic dependence on coordinate z for any distributions of field amplitude and phase[18]:

$$\sigma^2(z) = \sigma_0^{\,2} + \theta^2 z^2, \tag{9}$$

where θ has a sense of beam divergence and is related to the M^2 parameter by the formula

$$\theta = \frac{M^2}{k\sigma_0}. \tag{10}$$

For a Gaussian beam with a flat phase $M^2 = 1$, and formula (9) describes the changes in the beam radius along the z axis. Therefore formulas (9) and (10) yield the physical meaning of the M^2 parameter: It shows how many times the divergence of a real beam is greater than that of a Gaussian beam if the beams have the same effective radius in the waist. Note that the frequently used beam parameter product is proportional to M^2 and does not have its own meaning.

To compare the quality of the distorted ($M_{out}^{\,2}$) and the undistorted ($M_{in}^{\,2}$) beams let us introduce parameter μ:

$$\mu = \frac{M_{in}^{\,4}}{M_{out}^{\,4}}. \tag{11}$$

Parameter μ shows how many times the square of the beam divergence θ in formula (10) at the input is less than that at the output. It is the square of the divergence that is considered because we want to estimate the effective area of the beam and hence its mean intensity. In this paper we will analyze the phase distorted beams assuming that an undistorted (reference) beam has a flattop phase. In that case the parameter μ cannot be larger than 1, and it can reach 1 if and only if there are no distortions at all. We will proove this property of the parameter μ in Subsection 3.C. Note that both χ and S have this property as well.

Those three parameters (χ, S, μ) are mutually independent. It is necessary to find the interrelations between the parameters. In this paper we calculate (analytically and numerically) and optimize these criteria for Gaussian and super-Gaussian beam intensity profiles and for three particular types of phase distortions. In these particular cases we find regularities and relations between the criteria.

In Section 2 we will discuss three types of phase aberration: thermal lens, electronic self-focusing, and spherical aberrations. In Section 3 we derive expressions for all three criteria for all three types of distortion. The discussion in Section 4 consists of four main parts: (1) the validity of approximate analytical results, (2) a comparison of compensating lenses for maximizing the overlapping integral and Strehl ratios, (3) the influence of the beam shape on quality criteria, and (4) the comparison of different criteria.

2. Setting Up the Problem

It is reasonable to compare the criteria for concrete cases. We will consider distortions of only Gaussian and super-Gaussian beams with a flat phase:

$$E_{in}(r) = E \exp\left(-\frac{r^{2m}}{2a^{2m}}\right). \tag{12}$$

We will discuss only the three most common causes of beam quality degradation: thermal distortions (thermal isotropic lens), electronic self-focusing, and spherical aberrations.

The effect of the thermal lens consists of the fact that part of the power releases as heat attributable to finite absorption in optical elements. The absorption coefficient α is of the order of 10^{-3}–10^{-5} cm^{-1}. Thus at a power of 10 kW, 1 W is released on 1 cm of the length of the optical element. The intensity does not change greatly, but the optical element will be heated significantly. A temperature gradient occurs in the optical element. The dependence of the refractive index n on temperature T, and the photoelastic effect will be manifest. For cylinder geometry, phase distortions are defined by the formula[1]

$$\varphi_1(r) = -p_i \frac{m}{2\Gamma(1/m)} \int_0^{r^2/a^2} \left[\int_0^z \exp(-t^m) \frac{\mathrm{d}t}{z} \right] \mathrm{d}z = -p_i u_t(t), \tag{13}$$

where

$$p_i = \frac{P_L L \alpha P}{\lambda \kappa},$$

$$P = \frac{\mathrm{d}n}{\mathrm{d}T} - \frac{1}{L}\frac{\mathrm{d}L}{\mathrm{d}T}\frac{{n_0}^3}{4}\frac{1+v}{1-v}(p_{11}+p_{12})$$

$$t = r^2/a^2, \tag{14}$$

and P_L is the average power of radiation; κ, L, v, and p_{ij} are thermal conductivity, length, Poisson's ratio, and photoelastic coefficients of the optical element, respectively; and $\Gamma(x)$ is the Euler gamma function $m\int_0^\infty t^k \exp(-t^m)\mathrm{d}t = \Gamma((k+1)/m)$.

The effect of self-focusing consists in that at high electric fields the dependence of the refractive index on the electric field strength becomes apparent. In an isotropic medium, the phase distortions are defined by the formula[26]

$$\varphi_e(r) = B \exp\left(-\frac{r^{2m}}{a^{2m}}\right) = -Bu_e(t), \tag{15}$$

where

$$B = k\gamma \int_0^L I_0(z)\mathrm{d}z \tag{16}$$

is the B integral, I_0 is the intensity on the beam axis, and γ is the nonlinear refractive index. In contrast to the thermal lens, the peak intensity of the radiation plays an essential role in self-focusing. For optical elements this effect becomes strongly manifest at peak intensities of approximately 1 GW/cm^2.

Spherical aberrations are independent of power and come from lenses and mirrors with spherical surfaces. For small beam sizes and long-focus optics this can be neglected. In practice, spherical aberrations are irrelevant if the ratio of the beam diameter to the focal length is not more than 1/20, but this situation is not frequent, especially in high-energy lasers. In the first approximation, the distortions introduced in the phase by a telescope comprising two centered cofocal lenses are determined by[5,27]

$$\varphi_s(r) = -V\frac{r^4}{a^4} = -Vu_s(t), \tag{17}$$

where

$$V = \frac{ka^4}{2{f_2}^4}(G_1 f_1 + G_2 f_2), \tag{18}$$

and $f_{1,2}$ and $G_{1,2}$ are the focal lengths and the Seidel sums[27] of the telescope lenses. Thus for all three cases the output field has the form

$$E_{out}(r) = E\exp\left(-\frac{r^{2m}}{2a^{2m}}\right)\exp[-iqu(t)], \tag{19}$$

where q corresponds to p_i, B, or V from formulas (14), (16), and (18), and $u(t)$ is found from formulas (13), (15), and (17).

The phase distributions of the laser beam in the optical element are often near parabolic; thus a major part of the phase distortions can be compensated by a standard lens with a focal length F, which would introduce additional curvature in the wavefront. This is equivalent to multiplying the field E_{out} by a phase factor

$$\exp\left(\pm ik\frac{r^2}{2F}\right) = \exp\left[\pm iR\left(\frac{r}{a}\right)^2\right], \tag{20}$$

where

$$R = \frac{ka^2}{2F}. \tag{21}$$

By varying R the phase distortions can be minimized. We will denote the parameter χ and s improved in such a way as χ_{max} and S_{max}. It is worth noting that the parabolic phase corrector expression (20) does not change the M^2 parameter (and hence μ) at all.[18,24]

3. Calculating Quality Criteria

A. Calculating the Overlap Integral χ

We will use the general formula (19) for the field at the output, and then we will specify the results obtained for thermal distortions of self-focusing and spherical aberrations by replacing $u(t)$ by $u_t(t)$, $u_e(t)$, or $u_s(t)$ and q by p_i, B, or V. Substituting formulas (12) and (19) into formula (1) yields

$$\chi = \frac{m^2}{\Gamma(1/m)^2}\left|\int_0^\infty \exp(-t^m)\exp(-iqu(t))\mathrm{d}t\right|^2. \tag{22}$$

Table 1. Values of Series Expansion Coefficients of Quality Criteria χ, S, and μ at Various m[a]

Type of Distortion	Parameter of Distortion	Coefficient	$m = 1$	$m = 2$	$m = 4$	$m = 8$	$m = 16$	$m = \infty$
Thermal lens	p_i	$a_{1t}(m)$	0.067	0.039	0.028	0.023	0.021	1/3
		$a_{2t}(m)$	4.41×10^{-3}	5.13×10^{-4}	4.23×10^{-5}	2.47×10^{-6}	1.1×10^{-7}	0
		$\xi_t(m)$	0.25	0.463	0.528	0.527	0.516	1/2
		$b_{1t}(m)$	0.127	0.062	0.037	0.027	0.023	1/3
		$b_{2t}(m)$	0.016	1.77×10^{-3}	1.41×10^{-4}	7.96×10^{-6}	3.5×10^{-7}	0
		$\eta_t(m)$	0.167	0.41	0.512	0.524	0.516	1/2
		$c_t(m)$	0.038	7.21×10^{-3}	8.76×10^{-4}	8×10^{-5}	6.13×10^{-6}	0
Electron self-focusing	B	$a_{1e}(m)$	0.083	0.077	0.053	0.031	0.017	0
		$a_{2e}(m)$	0.021	2.2×10^{-3}	9.52×10^{-3}	0.015	0.012	0
		$\xi_e(m)$	0.25	0.643	0.661	0.441	0.245	0
		$b_{1e}(m)$	0.089	0.114	0.091	0.058	0.033	0
		$b_{2e}(m)$	0.04	9.56×10^{-3}	0.011	0.023	0.021	0
		$\eta_e(m)$	0.111	0.536	0.758	0.594	0.358	0
		$c_e(m)$	0.194	0.052	0.117	0.239	0.33	4/9
Spherical aberrations	V	$a_{1s}(m)$	20	0.5	0.136	0.091	0.084	4/45
		$a_{2s}(m)$	4	0.062	0.012	6.75×10^{-3}	5.61×10^{-3}	0
		$\xi_s(m)$	4	1.553	1.117	1.01	0.988	1
		$b_{1s}(m)$	320	2	0.272	0.129	0.1	4/45
		$b_{2s}(m)$	64	0.25	0.024	9.55×10^{-3}	6.67×10^{-3}	1/180
		$\eta_s(m)$	8	2.2	1.328	1.101	1.032	1
		$c_s(m)$	32	0.86	0.153	0.056	0.026	0

[a]Overlapping integral χ without compensation $a_1(m)$, formula (22), and with compensation $a_2(m)$, formula (28); Strehl ratio S without compensation $b_1(m)$, formula (34), and with compensation $b_2(m)$, formula (38); relation μ $c(m)$, formula (43). Values of $\xi(m)$, formula (26), and $\eta(m)$, formula (36).

In the general case, this integral cannot be calculated analytically. For small distortions, i.e., at $|qu(t)| \ll 1$, an expansion into the Taylor series in terms of a small parameter can be made. This approximation will always take place at small values of q because the function $u(t)$ is slower than $\exp(t^m)$. Limiting ourselves by a quadratic term and substituting into formula (22) expressions for $u(t)$ from formulas (13), (15), and (17), we obtain

$$\chi_0 = \chi(q \to 0) = 1 - q^2 a_1(m), \tag{23}$$

where

$$a_{1t}(m) = \frac{m}{\Gamma(1/m)} \int_0^\infty \exp(-t^m) u_t^2(t) \mathrm{d}t - \frac{m^2}{\Gamma(1/m)^2} \left[\int_0^\infty \exp(-t^m) u_t(t) \mathrm{d}t \right]^2,$$

$$a_{1e}(m) = 3^{-1/m} - 4^{-1/m},$$

$$a_{1s}(m) = \frac{\Gamma(5/m)}{\Gamma(1/m)} - \frac{\Gamma(3/m)^2}{\Gamma(1/m)^2}. \tag{24}$$

Here and in the rest of this paper the indices t, e, and s indicate the thermal lensing, self-focusing, and spherical aberrations, respectively. The numerical values of a_1 and subsequent approximation coefficients for each case are summarized in Table 1.

In the general case, optimization by means of phase corrector (20) can be solved only numerically. However, expanding into the Taylor series already in terms of small parameter $|tR - qu(t)|$ yields an optimal value of R_{opt} and a maximum value χ_{max}:

$$R_{opt} = q\xi(m), \tag{25}$$

$$\chi_{0\,max} = \chi_{max}(q \to 0) = 1 - q^2 a_2(m), \tag{26}$$

where

$$\xi_t(m) = -\frac{m\Gamma(2/m)}{\Gamma(1/m)\Gamma(3/m) - \Gamma(2/m)^2} \times \int_0^\infty u_t(t) \left[1 - t \frac{\Gamma(1/m)}{\Gamma(2/m)} \right] \exp(-t^m) \mathrm{d}t,$$

$$\xi_e(m) = \frac{\Gamma(2/m)\Gamma(1/m)}{(\Gamma(1/m)\Gamma(3/m) - \Gamma(2/m)^2)} [2^{-1/m} - 4^{-1/m}],$$

$$\xi_s(m) = -\frac{\Gamma(2/m)}{\Gamma(1/m)\Gamma(3/m) - \Gamma(2/m)^2} \times \left[\Gamma(3/m) - \frac{\Gamma(1/m)}{\Gamma(2/m)} \Gamma(4/m) \right], \tag{27}$$

$$a_{2t}(m) = a_{1t}(m) - \frac{m^2\Gamma(2/m)^2}{\Gamma(1/m)^2(\Gamma(1/m)\Gamma(3/m) - \Gamma(2/m)^2)} \times \left\{\int_0^\infty u_t(t)\left[1 - t\,\frac{\Gamma(1/m)}{\Gamma(2/m)}\right]\exp(-t^m)\mathrm{d}t\right\}^2,$$

$$a_{2e}(m) = a_{1e}(m) - \frac{\Gamma(2/m)^2}{\Gamma(1/m)\Gamma(3/m) - \Gamma(2/m)^2} \times (2^{-1/m} - 4^{-1/m})^2,$$

$$a_{2s}(m) = a_{1s}(m) - \frac{\Gamma(2/m)^2}{[\Gamma(1/m)\Gamma(3/m) - \Gamma(2/m)^2]} \times \left[\Gamma(3/m) - \frac{\Gamma(1/m)}{\Gamma(2/m)}\,\Gamma(4/m)\right]^2. \tag{28}$$

Formulas (24) and (28) for coefficients $a_{1t}(m)$ and $a_{2t}(m)$ coincide with the result obtained earlier in this paper.[1] Note that for a Gaussian beam ($m = 1$) formulas (24) and (28) for coefficients $a_{1t}(m)$ and $a_{2t}(m)$ will be significantly simplified, and

$$R_{opt}(m = 1) = \frac{p_i}{4}. \tag{29}$$

B. Calculating the Strehl Ratio S

To calculate the Strehl ratio by definition (6), the field distribution in the focal plane of the aberrationless lens should be found. To do this it is necessary to perform the spatial Fourier transform. It can be easily obtained that

$$S = \frac{\left|\iint_\Omega E_{out}\mathrm{d}S\right|^2}{\left|\iint_\Omega E_{in}\mathrm{d}S\right|^2}. \tag{30}$$

Substituting formulas (12) and (19) into formula (30) yields

$$S = \frac{m^2}{\Gamma(1/m)^2}\left|\int_0^\infty \exp(-t^m)\exp\left(-iqu\left(t\sqrt[m]{2}\right)\mathrm{d}t\right)\right|^2. \tag{31}$$

A comparison of formula (31) with expression (22) readily shows that χ and S differ only by arguments to function $u(t)$. Moreover, for a flattop beam ($m = \infty$) there is no difference at all. This fact makes the process of studying the Strehl ratio S much easier. Expanding this expression into the Taylor series in terms of small parameter $|qu(t\sqrt[m]{2})|$ we obtain

$$S_0 = S(q \to 0) = 1 - q^2 b_1(m), \tag{32}$$

where

$$b_{1t}(m) = \frac{m}{\Gamma(1/m)}\int_0^\infty \exp(-t^m)u_t^2\left(t\sqrt[m]{2}\right)\mathrm{d}t - \frac{m^2}{\Gamma(1/m)^2} \times \left[\int_0^\infty \exp(-t^m)u_t\left(t\sqrt[m]{2}\right)\mathrm{d}t\right]^2,$$

$$b_{1e}(m) = 5^{-1/m} - 9^{-1/m},$$

$$b_{1s}(m) = 16^{1/m}a_{1s}(m) = 16^{1/m}\left[\frac{\Gamma(5/m)}{\Gamma(1/m)} - \frac{\Gamma(3/m)^2}{\Gamma(1/m)^2}\right]. \tag{33}$$

Similarly, we can increase the number S by phase corrector expression (20). Expanding into the Taylor series already in terms of small parameter $|\sqrt[m]{2}tR - qu(\sqrt[m]{2}t)|$ yields an optimal value of R_{opt} and maximum value of S_{max}:

$$R_{opt} = q\eta(m), \tag{34}$$

$$S_{0\,max} = S_{max}(q \to 0) = 1 - q^2 b_2(m), \tag{35}$$

where

$$\eta_t(m) = -\frac{1}{\sqrt[m]{2}}\,\frac{m\Gamma(2/m)}{\Gamma(1/m)\Gamma(3/m) - \Gamma(2/m)^2} \times \int_0^\infty u_t\left(t\sqrt[m]{2}\right)\left[1 - t\,\frac{\Gamma(1/m)}{\Gamma(2/m)}\right]\exp(-t^m)\mathrm{d}t,$$

$$\eta_e(m) = \frac{1}{\sqrt[m]{2}}\,\frac{\Gamma(2/m)\Gamma(1/m)}{\Gamma(1/m)\Gamma(3/m) - \Gamma(2/m)^2}(3^{-1/m} - 9^{-1/m}),$$

$$\eta_s(m) = \sqrt[m]{2}\xi_s(m) = -\sqrt[m]{2}\,\frac{\Gamma(2/m)}{\Gamma(1/m)\Gamma(3/m) - \Gamma(2/m)^2} \times \left[\Gamma(3/m) - \frac{\Gamma(1/m)}{\Gamma(2/m)}\,\Gamma(4/m)\right], \tag{36}$$

$$b_{2t}(m) = b_{1t}(m) - \frac{m^2\Gamma(2/m)^2}{\Gamma(1/m)^2[\Gamma(1/m)\Gamma(3/m) - \Gamma(2/m)^2]} \times \left\{\int_0^\infty u_t\left(t\sqrt[m]{2}\right)\left[1 - t\,\frac{\Gamma(1/m)}{\Gamma(2/m)}\right]\exp(-t^m)\mathrm{d}t\right\}^2,$$

$$b_{2e}(m) = b_{1e} - \frac{\Gamma(2/m)^2}{\Gamma(1/m)\Gamma(3/m) - \Gamma(2/m)^2}(2^{-1/m} - 4^{-1/m})^2,$$

$$b_{2s}(m) = 16^{1/m}a_{2s}(m) = b_{1s}(m) - \frac{16^{1/m}\Gamma(2/m)^2}{[\Gamma(1/m)\Gamma(3/m) - \Gamma(2/m)^2]} \times \left[\Gamma(3/m) - \frac{\Gamma(1/m)}{\Gamma(2/m)}\,\Gamma(4/m)\right]^2. \tag{37}$$

For a Gaussian beam, one can find

$$R_{opt} = \frac{p_i}{6}. \tag{38}$$

As seen from formulas (29) and (38) the compensating focus lens from the Strehl ratio viewpoint is longer by a factor of 1.5 than from the overlapping integral viewpoint.

C. Calculating the M^2 Parameter

Using definition (7), the M^2 parameter can easily be calculated for the case of a flat phase. In reality, phase additions shift the waist along the z axis of beam propagation. Thus, to find the waist's position, we need to know the value of the M^2 parameter. The calculation procedure of M^2 can be much simplified by applying the method of moments.[28,29] As was previously shown in Ref. 24, the method gives the following expression for M^2:

$$M^2 = k\sqrt{A\sigma_0{}^2 - b^2/4}, \tag{39}$$

where

$$A = \frac{1}{k^2}\frac{\int_0^\infty (\nabla E)^2 r\,\mathrm{d}r}{\int_0^\infty E^2 r\,\mathrm{d}r} + \frac{\int_0^\infty (\nabla\varphi/k)^2 E^2 r\,\mathrm{d}r}{\int_0^\infty E^2 r\,\mathrm{d}r}, \tag{40}$$

$$b = \frac{2}{k}\frac{\int_0^\infty (r\nabla\varphi)E^2 r\,\mathrm{d}r}{\int_0^\infty E^2 r\,\mathrm{d}r}. \tag{41}$$

As we mentioned after Eq. (21), in contrast to χ and S, the M^2 parameter does not change by the phase corrector expression (20).[18,24] It can be directly proved by means of expressions (39)–(41). Substituting $\varphi_{corr} = \varphi + const \cdot r^2$ instead of φ in formula (41), one can choose the constant that provides $b(\nabla\varphi_{corr}) = 0$. Because the parameter M^2 does not depend on the phase corrector expression (20), formula (39) reduces to

$$M^2 = k\sigma_0\sqrt{A(\varphi_{corr})}. \tag{42}$$

Formulas (39) and (42) show that the M^2 parameter has its minimum value if and only if $\varphi_{corr} = const$, i.e., for a nonaberrated beam (a beam with a plane or pure parabolic phase). Hence parameter $\mu = 1$ only for any nonaberrated beam, and $\mu < 1$ for any aberrated beam. This property enables μ (as well as χ and S) to act as a quality criterion of phase-aberrated beams.

Substituting expressions (12), (13), (15), and (17) into expressions (39)–(41), and the results to expression (11), we obtain

$$\mu = \frac{1}{1 + q^2 c(m)}, \tag{43}$$

where we have for the thermal distortions, self-focusing, and spherical aberrations, respectively,

$$c_t(m) = \frac{m}{\Gamma(1/m)^2}\int_0^\infty \left[\int_0^t \exp(-y^m)\mathrm{d}y\right]^2 \exp(-t^m)\frac{\mathrm{d}t}{t} - \frac{1}{4}\frac{\Gamma(1/m)^2}{m^2\Gamma(2/m)},$$

$$c_e(m) = \frac{4}{9} - 4^{-1/m}\frac{\Gamma(1/m)^2}{m^2\Gamma(2/m)},$$

$$c_s(m) = \frac{16}{m^2\Gamma(2/m)}(\Gamma(2/m)\Gamma(4/m) - \Gamma(3/m)^2). \tag{44}$$

For comparison with other quality criteria, we expand this expression into the Taylor series in terms of small parameter $q^2c(m)$ to obtain

$$\mu \approx \mu_0 = \mu(q \to 0) = 1 - q^2 c(m). \tag{45}$$

4. Discussion

An analysis of the results shows that for a wide range of parameters the type of distortion does not dramatically affect the quality criteria χ, S, and μ. There are, however, some differences, and these will be discussed below. Here, as above, argument q will be one of the parameters p_i, B, or V, characterizing the relative types of distortion.

A. Comparison of Numerical and Analytical Results

First of all, let us discuss how accurately the approximated formulas (27) for R_{opt} describe the exact values of R_{ex} obtained numerically. The dependences $R_{opt}(q)$ and $R_{ex}(q)$ are presented in Figs. 1–3, showing their good coincidence in a fairly large area.

Figures 4–6 illustrate the corresponding plots $\chi_{max}(q)$ and $\chi(q)$, both exact and approximated. It is evident that at small q the exact dependences are well approximated by the obtained theoretical parabolas, and when the distortions are compensated the approximation is performed in a quite wider range of q. The corresponding dependences for the Strehl ratio S are qualitatively similar to those for χ.

As was said earlier, an essential feature of the M^2 parameter is that it does not change when an aberrationless lens is placed at the input. Thus we cannot improve this parameter in such a way. Moreover the analytical expression for $c(m)$ allowed us to calculate both μ and μ_0.

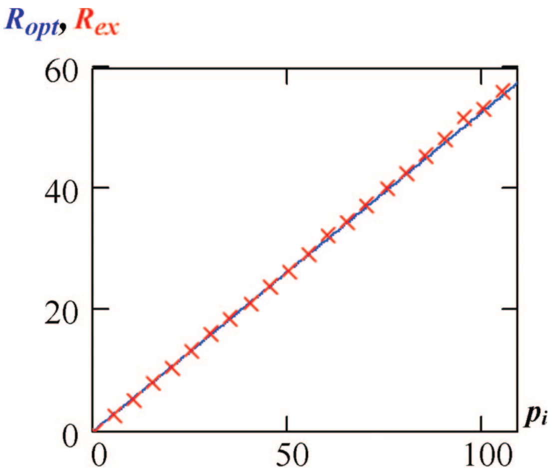

Fig. 1. (Color online) Dependences $R_{opt}(p_i)$ (solid line) and $R_{ex}(p_i)$ (pluses) after optimization of overlapping integral χ^5 at $m = 4$.

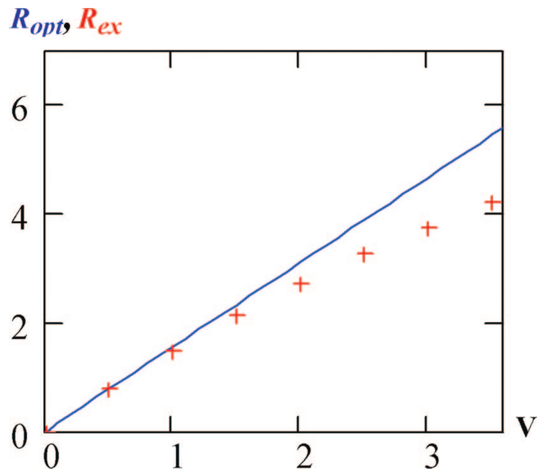

Fig. 3. (Color online) Dependences $R_{opt}(V)$ (solid line) and $R_{ex}(V)$ (pluses) after optimization of overlapping integral χ at $m = 2$.

B. Comparison of Optimal Compensation Lenses

Both χ and S may be maximized by an aberrationless lens [expression (20)]. But optimizing values of the focal length differ from the viewpoints of χ and S. Let us compare the parameter R_{opt}, which is inversely proportional to the compensating lens focal length [see expression (21)] for different types of aberration, with R_{ex}. As seen from Table 1 and Fig. 7 the parameter R_{opt} is larger for overlapping integral χ (ξ_t always larger than η_t) at thermal lensing and smaller for Strehl ratio S (ξ_s always smaller than η_s) at spherical aberrations. For both types of aberration the highest difference takes place for a Gaussian beam: by a factor of 1.5 for thermal lensing and by a factor of 2 for spherical aberration.

At self-focusing both cases are possible depending on the beam shape: Parameter R_{opt} is greater for χ and for S. For a Gaussian beam R_{opt} is greater for an overlapping integral ($\xi_e = 2.25\eta_e$), but for a super-Gaussian beam with $m = 16$, R_{opt} is greater for the Strehl ratio ($\xi_e = 0.68\eta_e$). Thus, the choice of compensation lens strongly depends on the beam shape and the quality criteria.

C. Dependence of Quality Criteria on Beam Shape

Now we will consider in more detail the dependence of the beam quality criteria on the beam shape, which is characterized by the parameter m. In doing this, we will use the dependences of the quality criteria expansion coefficients on m. In the case of thermal distortions, the behavior of the dependences is identical: all five coefficients [$a_{1t}(m)$, $b_{1t}(m)$, $a_{2t}(m)$, $b_{2t}(m)$, $c_t(m)$] monotonically decrease with increasing m over the whole range of values (Fig. 8). At $m \to \infty$ (flattop beam) $a_{1t}(m)$ and $b_{1t}(m)$ tend to constants and $a_{2t}(m)$, $b_{2t}(m)$, $c_t(m)$ to zero (Table 1). In other words, all three criteria χ, S, and μ tend to their highest possible value, which is equal to unity. For the flattop beam, phase distortions are parabolic, so they can be totally compensated by an ideal lens.

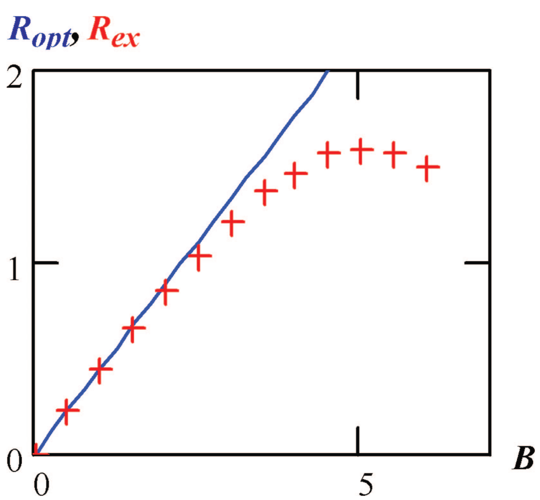

Fig. 2. (Color online) Dependences $R_{opt}(B)$ (solid line) and $R_{ext}(B)$ (pluses) after optimization of overlapping integral χ at $m = 8$.

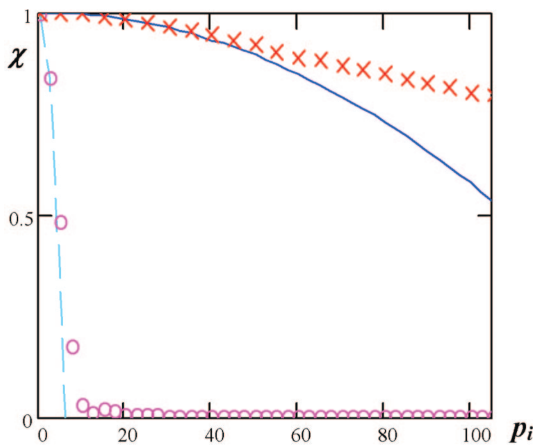

Fig. 4. (Color online) Dependences of the overlapping integral χ on parameter p_i at $m = 4$: without compensation, plotted by formula (23) $\chi_0(p_i)$ (dashed line) and numerically $\chi(p_i)$ (circles); with compensation, plotted numerically $\chi_{max}(p_i, R_{ex})$, (crosses), by approximated formula (26) $\chi_{0\,max}(p_i, R_{opt})$ (solid curve).

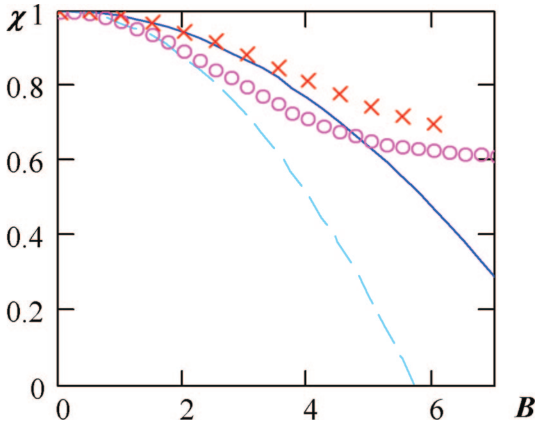

Fig. 5. (Color online) Dependences of the overlapping integral χ on parameter B at $m = 8$: without compensation, plotted by formula (23) $\chi_0(B)$ (dashed curve) and numerically $\chi(B)$ (circles); with compensation, plotted numerically $\chi_{max}(B, R_{ex})$ (crosses), by approximated formula (26) $\chi_{0\,max}(B, R_{opt})$ (solid curve).

For self-focusing the results are notably different (Fig. 9). First, at $m \to \infty$ coefficients $a_{1e}(\infty) = 0$ and $b_{1e}(\infty) = 0$, and starting from $m = 2$, the decrease becomes monotonic. The dependences $a_{2e}(m)$ and $b_{2e}(m)$ have one local minimum and maximum each in close but yet at different points, and at m going to infinity they tend to zero. The dependence $c_e(m)$ also has a local minimum, but at $c_e(m \to \infty) = 4/9$.

From a physical viewpoint, there is no phenomenon of self-focusing for a flattop beam, and hence there should be no distortions and nothing to compensate. Thus the overlapping integral and the Strehl ratio are in good agreement with it, and their description is almost equivalent, while the μ criterion is inappropriate for describing the flattop beam. The reason for the nonphysical nature of the criterion μ for beams with $m \gg 1$ is that the M^2 parameter grows with

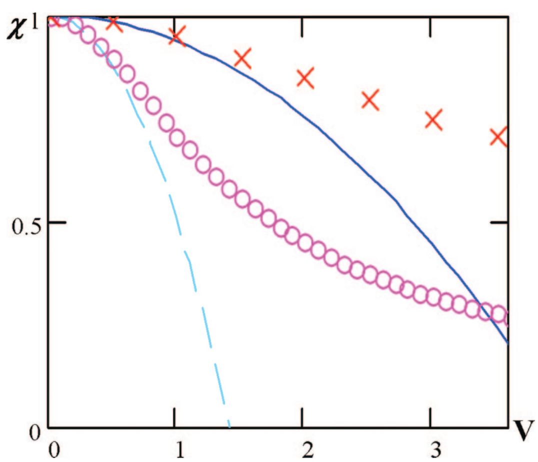

Fig. 6. (Color online) Dependences of the overlapping integral χ on parameter V at $m = 2$: without compensation, plotted by formula (23) $\chi_0(V)$ (dashed curve) and numerically $\chi(V)$ (circles); with compensation, plotted numerically $\chi_{max}(V, R_{ex})$ (crosses), by approximated formula (26) $\chi_{0\,max}(V, R_{opt})$ (solid curve).

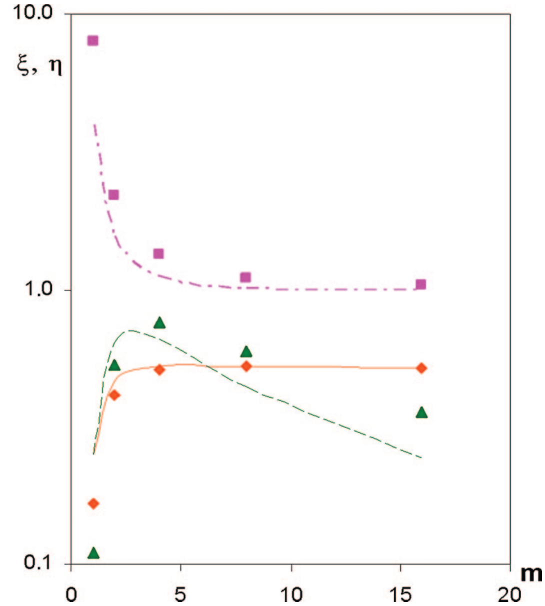

Fig. 7. (Color online) Dependences of series expansion coefficients of R_{opt} on parameter m for thermal distortions $\xi_t(m)$ (solid curve), $\eta_t(m)$ (diamonds); electronic self-focusing $\xi_e(m)$ (dashed curve), $\eta_e(m)$ (triangles); spherical aberrations $\xi_s(m)$ (dashed–dotted curve), $\eta_s(m)$ (boxes); plotted according to formulas (27) and (36).

growing m and even tends to infinity at $m \to \infty$.[21,24] Thus the relation μ also loses its physical meaning.

The situation with spherical aberrations is similar to thermal distortions (Fig. 10). The only difference is that $a_{1s}(m \to \infty) = b_{1s}(m \to \infty) = 1/16$ and $c_s(m \to \infty) = 0$. Since the phase is proportional to the fourth power of the radius, and the lens gives a quadratic addition to the phase, the distortions cannot be totally compensated. Thus the overlapping integral χ and the Strehl ratio S provide almost identical descriptions. As for $c_s(m \to \infty) = 0$, it indicates that again the param-

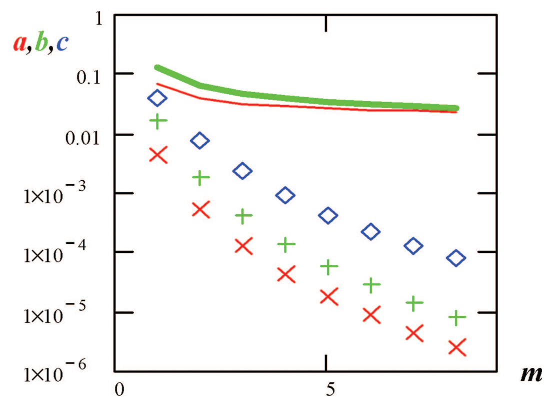

Fig. 8. (Color online) Dependences of series expansion coefficients of quality criteria χ, S, and μ on parameter m for thermal distortions: overlapping integral χ without compensation $a_{1t}(m)$ (solid curve), formula (24), and with compensation $a_{2t}(m)$ (crosses), formula (28); Strehl ratio S without compensation $b_{1t}(m)$ (bold solid curve), formula (33), and with compensation $b_{2t}(m)$ (pluses), formula (37); relation μ $c_t(m)$ (diamonds), formula (40).

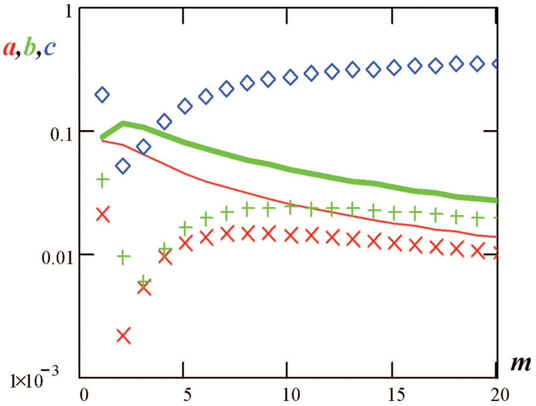

Fig. 9. (Color online) Dependences of series expansion coefficients of quality criteria χ, S, and μ on parameter m for electronic self-focusing: overlapping integral χ without compensation $a_{1e}(m)$ (solid curve), formula (24), and with compensation $a_{2e}(m)$ (crosses), formula (28); Strehl ratio S without compensation $b_{1e}(m)$ (bold solid curve), formula (33), and with compensation $b_{2e}(m)$ (pluses), formula (37); relation μ $c_e(m)$ (diamonds), formula (40).

eter μ describes the flattop beam ($m \to \infty$) nonphysically because $M^2 \to \infty$.[21,24]

D. Comparison of Different Criteria

The dependences of all three criteria χ, S, and μ on parameter q are shown in Figs. 11–13 in both cases: without compensation and with compensation by lens with optimal focal length. As we mentioned above, the parameter μ does not depend on the lens. Let us first compare noncompensated criteria. To compare different criteria we will also use Figs. 8–10, where series expansion coefficients are shown: the smaller the coefficient, the larger the corresponding quality criterion. Without compensation there is only one common relation, which is valid for any Gaussian and super-Gaussian beam at all three types of distortion: $\chi > S$.

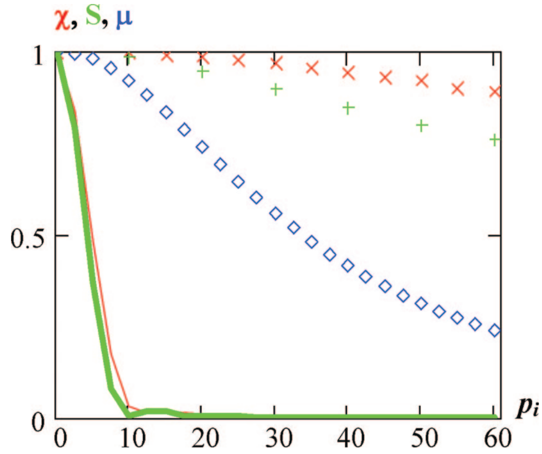

Fig. 11. (Color online) Numerical dependences of quality criteria χ, S, and μ on parameter p_i at $m = 4$: overlapping integral χ without compensation $\chi(p_i)$ (thin curve) and with compensation $\chi_{max}(p_i, R_{ex})$ (crosses); Strehl ratio S without compensation $S(p_i)$ (thick curve) and with compensation $S_{max}(p_i, R_{ex})$ (pluses); relation $\mu(p_i)$ (diamonds).

A comparison of compensated criteria $\chi_{max}(q)$, $S_{max}(q)$, and $\mu(q)$ is interesting because such correction is simple. Figures 11–13 show that graph $\mu(q)$ is always below, $S_{max}(q)$ is in the middle, and $\chi_{max}(q)$ is above. This fact can be once more corroborated by the dependences of the expansion coefficients $a_2(m)$, $b_2(m)$, and $c(m)$ for any type of distortion we considered here [in the case of spherical aberrations for parameters χ and S this is evident directly from formula (33)]. Namely, at $m \geq 1$ the inequality $a_2(m) < b_2(m) < c(m)$ is always fulfilled with only one exception, $a_{2s}(1) < c_s(1) < b_{2s} < (1)$ (see Table 1). Hence the criterion μ is the most rigid, and χ is the least rigid in all considered cases except for a Gaussian beam with spherical aberrations.

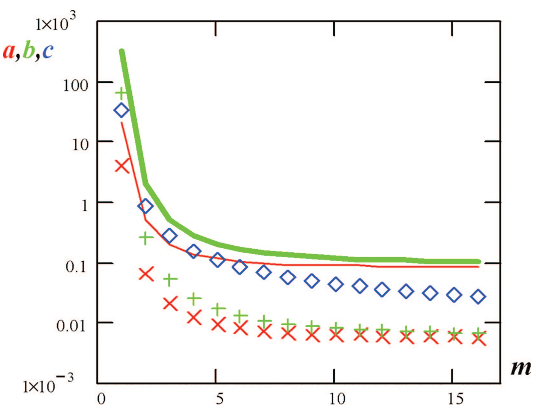

Fig. 10. (Color online) Dependences of series expansion coefficients of quality criteria χ, S, and μ on parameter m for spherical aberrations: overlapping integral χ without compensation $a_{1s}(m)$ (solid curve), formula (24), and with compensation $a_{2s}(m)$ (crosses), formula (28); Strehl ratio S without compensation $b_{1s}(m)$ (bold solid curve), formula (33), and with compensation $b_{2s}(m)$ (pluses), formula (37); relation μ $c_s(m)$ (diamonds), formula (40).

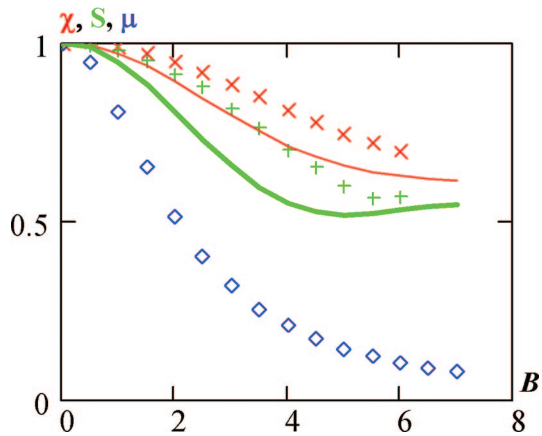

Fig. 12. (Color online) Numerical dependences of quality criteria χ, S, and μ on parameter B at $m = 8$: overlapping integral χ without compensation $\chi(B)$ (thin curve) and with compensation $\chi_{max}(B, R_{ex})$ (crosses); Strehl ratio S without compensation $S(B)$ (thick curve) and with compensation $S_{max}(B, R_{ex})$ (pluses); relation $\mu(B)$ (diamonds).

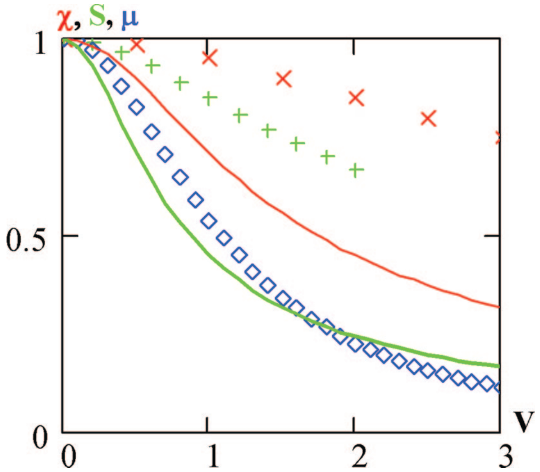

Fig. 13. (Color online) Numerical dependences of quality criteria χ, S, and μ on parameter V at $m = 2$: overlapping integral χ without compensation $\chi(V)$ (thin curve) and with compensation $\chi_{max}(V, R_{ex})$ (crosses); Strehl ratio S without compensation $S(V)$ (thick curve) and with compensation $S_{max}(V, R_{ex})$ (pluses); relation $\mu(V)$ (diamonds).

The comparison of criteria χ, S, and μ gives another interesting result. Let us remember the physical meaning of parameter χ. Let us represent the field at the output in the form of formula (2) and remove (filter out) component $E_0(r)$ from the field. Such a filtering may be done by a high-finesse cavity or by a device theoretically and experimentally studied in Ref. 30. There will be only $\widehat{E}_{out}(r) = NE_{in}(r)$ left, and according to formula (5), $|\widehat{E}_{out}(r)| = \sqrt{\chi}|E_{in}(r)|$. By substituting these expressions for the field $\widehat{E}_{out}(r)$ at the output into formulas (1), (30), and (7) for χ, S, and μ, respectively, it is easy to obtain that in this case χ and μ will be equal to 1 (radiation power has no influence upon χ and μ), and S will be equal to χ for a nonfiltered beam. Since χ is more than S (Figs. 11–13), we obtain that, as a result of filtering, the Strehl ratio S, and, correspondingly, the intensity on the beam axis in the focal plane always increase in spite of the decreasing beam power.

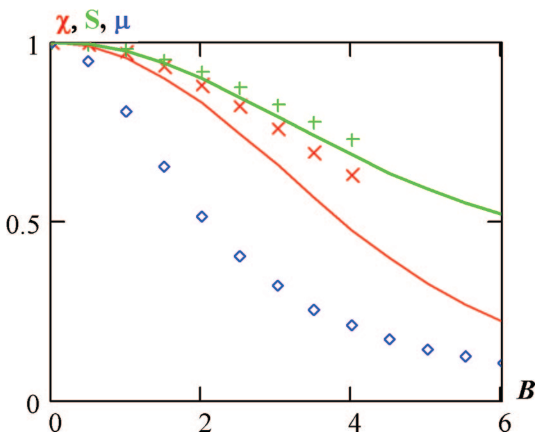

Fig. 14. (Color online) Numerical dependences of quality criteria χ, S, and μ on parameter B at $m = 0.5$: overlapping integral χ without compensation $\chi(B)$ (thin curve) and with compensation $\chi_{max}(B, R_{ex})$ (crosses); Strehl ratio S without compensation $S(B)$ (thick curve) and with compensation $S_{max}(B, R_{ex})$ (pluses); relation $\mu(B)$ (diamonds).

All that is said above is also valid for the case of $m < 1$ except for self-focusing. For example, $a_{2e}(m = 0.5) < b_{2e}(m = 0.5)$, and dependence $\chi_{max}(B)$ is lower than $S_{max}(B)$ (see Fig. 14).

5. Conclusion

In this study the three criteria most frequently used to quantitatively describe beam distortions were compared for thermal distortions, self-focusing, and the spherical aberrations of Gaussian and super-Gaussian beams.

Without compensation we found only one common relation, which is valid for any Gaussian and super-Gaussian beam at all three types of distortion: Overlapping integral χ is larger than Strehl ratio S.

The overlapping integral χ and Strehl ratio S can be increased in many cases by means of a standard lens, so it is reasonable to compare the improved (maximized) values. We show that the focal length of the compensating lens can be quite accurately calculated by the obtained approximated analytical formulas. The behavior of these two parameters is very similar. Specifically, the overlapping integral and the Strehl ratio decrease with increasing distortions and change identically depending on the beam shape. Nonetheless, we did not find any simple and unambiguous correspondence for their quantitative comparison.

The μ criterion is not suitable for the description of a flattop beam, and its behavior at changing m is different from that of the other two criteria. The cause is in the growth of the M^2 parameter for large m, with $M^2 \to \infty$ for the flattop beam, which suggests its nonphysical nature in this case.

The maximized overlapping integral χ is greater than the Strehl ratio S, and S is always greater than μ for any Gaussian or super-Gaussian beam and for all three types of distortion: thermal lens, electronic self-focusing, and spherical aberrations. In other words, the overlapping integral is the least rigid criterion, whereas μ is the most rigid one.

When part of the distorted beam is filtered out, the intensity on the beam's axis in the focal plane can be increased, although the total power of the beam will decrease at filtering. This statement is valid with and without compensation for all considered distortions and beam shapes.

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